Government College of Engineering Keonjhar

LECTURE NOTES

MATHS-II

VECTOR CALCULUS

Module - IV (10 Hours)

Syllabus: Vector integral calculus: Line Integrals, Green Theorem, Surface integrals, Volume integral, Gauss theorem and Stokes Theorem.

LINE INTEGRL :

Single integral as a function defined on a segment of a curve is called Line integral. Line integral of vector vector field $\vec{F}$ along some curve C can be written as

$$\int_C \vec{F}.d\vec{r} = \int_C F_1 dx + F_2 dy + F_3 dz$$

Where, $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$ *and* $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$

Evaluate

$$\int_C y^2 dx - x^2 dy$$

C: straight line segment from (0,0) to (1, 1).

Given integral $\int_C y^2 dx - x^2 dy$

C: straight line segment from (0,0) to (1, 1)

$$\frac{y-0}{1-0} = \frac{x-0}{1-0} = t$$

$$y = x = t$$

$$\Rightarrow dy = dx = dt$$

$$t \text{ vary from } t = 0 \text{ to } t = 1$$

$$\int_C y^2 dx - x^2 dy = \int_C t^2 dt - \int_C t^2 dt$$

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$$= \int_0^1 t^2 dt - \int_0^1 t^2 dt = \frac{1}{3} - \frac{1}{3} = 0$$

Example Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = x^2y^2\,\hat{i} + y\,\hat{j}$ and curve C is $y^2 = 4x$ in xy-plane from $(0,0)$ to $(4,4)$

Soln —

$$\vec{r} = x\hat{i} + y\hat{j}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

Given $\vec{F} = x^2y^2\,\hat{i} + y\,\hat{j}$

$$\vec{F} \cdot d\vec{r} = (x^2y^2\,\hat{i} + y\hat{j}) \cdot (dx\,\hat{i} + dy\,\hat{j})$$

$$= x^2y^2\,dx + y\,dy.$$

Given curve, $y^2 = 4x$ in xy plane from $(0,0)$ to $(4,4)$

~~For~~ So,

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (x^2y^2\,dx + y\,dy)$$

$$= \int_C x^2y^2\,dx + \int_C y\,dy$$

So, $\int \vec{F} \cdot d\vec{r} = \int_C x^2y^2\,dx + \int_C y\,dy$

$C: y^2 = 4x$ from $(0,0)$ to $(4,4)$

x vary from 0 to 4

y vary from 0 to 4

$$= \int_0^4 x^2(4x)\,dx + \int_0^4 y\,dy$$

$$= 4\int_0^4 x^3\,dx + \left[\frac{y^2}{2}\right]_0^4$$

$$= \left[x^4\right]_0^4 + \frac{16}{2}$$

$$= 256 + 8$$

$$= 264 \quad \text{Ans.}$$

Example Find work done in moving particle in a force field $\vec{F} = 3x^2\hat{i} + (2xz-y)\hat{j} + z\hat{k}$ along line joining $(0,0,0)$ to $(2,1,3)$.

Soln Work done $= \int_C \vec{F}\cdot d\vec{r}$

Given, $\vec{F} = 3x^2\hat{i} + (2xz-y)\hat{j} + z\hat{k}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

Work done $= \int_C [3x^2\hat{i} + (2xz-y)\hat{j} + z\hat{k}]\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})]$

$= \int_C (3x^2 dx + (2xz-y)dy + z\,dz)$

C: line joining $(0,0,0)$ to $(2,1,3)$

$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = t \Rightarrow$ $x = 2t \Rightarrow dx = 2dt$, $y = t \Rightarrow dy = dt$, $z = 3t \Rightarrow dz = 3dt$.

Work done $= \int_C [3x^2 dx + (2xz-y)dy + z\,dz]$

C: $x = 2t \Rightarrow dx = 2dt$ $\left(\begin{matrix} x=0 & x=2 \\ t=0 & t=1 \end{matrix}\right)$

$y = t \Rightarrow dy = dt$

$z = 3t \Rightarrow dz = 3dt$

Work done $= \int_0^1 [3(2t)^2(2dt) + (2.2t.3t - t)dt + 3t\,dt]$

$= \int_0^1 (36t^2 + 8t)\,dt$

$= \cancel{36}^{12}\left[\frac{t^3}{\cancel{3}}\right]_0^1 + \cancel{8}\left(\frac{t^2}{2}\right)_0^1$

$= 12 + 4$

$= 16.$

Conservative Field

Definition: A vector field $\vec{F}$ defined in some region is called conservative if

$$\int_{C_1} \vec{F}.d\vec{r} = \int_{C_2} \vec{F}.d\vec{r}$$

whenever C_1 and C_2 are any two simple curves in the region with the same initial and terminal points.

Note: 1) A vector field $\vec{F}$ is conservative (irrotational) if and only if $\int_C \vec{F}.d\vec{r} = 0$ 1for every simple closed curve in the region where $\vec{F}$ is defined.

2) If $\vec{F}$ is conservative, then $\vec{F}$ is necessarily gradient of some scalar function.

3) If a vector field F is defined in a simply-connected region in the xy-plane and ∇ × F = 0 throughout that region, then F is conservative.

Example

Show that $\vec{F} = (2xy + z^3)\hat{\imath} + x^2\hat{\jmath} + 3xz^2\hat{k}$ is conservative force field. Find scalar potential ϕ of $\vec{F}$.

Soln. : If Curl $\vec{F} = 0$

$$\vec{\nabla} \times \vec{F} = 0$$

Then $\vec{F}$ is called conservative field (irrotational vector).

$$Curl\,\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= \hat{\imath}(0 - 0) - \hat{\jmath}(3z^2 - 3z^2) + \hat{k}(2x - 2x)$$

$$= 0$$

Thus, $\vec{F}$ is called conservative force field.

Soln. : If $\vec{F}$ is conservative, then $\vec{F}$ is necessarily gradient of some scalar function. Thus,

$$\vec{F} = \nabla\phi$$

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$$(2xy + z^3)\hat{\imath} + x^2\hat{\jmath} + 3xz^2\hat{k} = \vec{F} = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$\frac{\partial\phi}{\partial x} = 2xy + z^3 \Rightarrow \phi = x^2y + xz^3 + f_1(y,z)$

$\frac{\partial\phi}{\partial y} = x^2 \Rightarrow \phi = x^2y + f_2(x,z)$

$\frac{\partial\phi}{\partial z} = 3xz^2 \Rightarrow \phi = xz^3 + f_3(x,y)$

All equations are same

Hence

$$f_1(y,z) = 0$$

$$f_2(x,z) = xz^3$$

$$f_3(x,y) = x^2y$$

Hence, $\phi = x^2y + xz^3$ **Ans.**

Area

1. **In calculus of a single variable y** $= f(x)$ **the definite integral**

$$\int_{x=a}^{x=b} f(x)dx$$

for $f(x) > 0$ **is area under the curve from** $x = a$ **to** $x = b$**.**

2. **In calculus of a two variable the definite integral z** $= f(x,y)$

$$\iint_S f(x,y)dxdy$$

   a) **If** $f(x,y) = 1$**, then**

$$\text{Area} = \iint_S dxdy$$

b) If $f(x,y) > 0$**, the definite integral is equal to the volume under the surface z** $= f(x,y)$ **and above** $x\ y$**-plane for x and y in the region R**

Area using line integral

Let C be simply connected smooth curve with anti clockwise direction in the plane R

$$Area = \frac{1}{2}\left(\int_C xdy - ydx\right)$$

Using green's theorem, $\int_C F_1(x,y)dx + F_2(x,y)dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$

$F_1 = -y, F_2 = x$

$$\int_C xdy - ydx = 2\iint_S dxdy = Area$$

This integral is usually evaluated with help of parametric form.

SURFACE INTEGRALS

SURFACE INTEGRAL ($\iint_S dxdy$) $\int dx$

An integral which is to be evaluated over a surface is called surface integral.

Let $\vec{F}$ be vector point function & let S be given surface

$\therefore$ Surface integral $= \iint_S \vec{F}\cdot\hat{n}\,ds$ or $\iint_S \vec{F}\cdot d\vec{S}$

where

$\hat{n}$ = unit normal vector to an element ds

$d\vec{S}$ = area of element ds

Note –

1. If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ $\quad d\vec{S} = dydz\,\hat{i} + dxdz\,\hat{j} + dxdy\,\hat{k}$

$S.I. = \iint \vec{F}\cdot d\vec{S} = \iint (F_1 dydz + F_2 dxdz + F_3 dxdy)$

2. Let R_1 be projection on surface S on xy-plane then,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{R_1} \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\hat{k}|}\,dxdy \quad \left\{ \text{i.e. } ds = \frac{dxdy}{|\hat{n}\cdot\hat{k}|} \right\}$$

3. If projection of surface S in xy-plane which is circle $x^2+y^2=a^2$, then, use polar co-ordinate (r,θ) for solving integration

Put $x = r\cos\theta$, $y = r\sin\theta$, $\boxed{dxdy = r\,dr\,d\theta}$

4. If S is surface of the shape ϕ, then normal vector $\vec{n} = \text{grad}\,\phi$ and $\hat{n} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$

Example Evaluate $\iint \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ and S is that part of surface of the sphere $x^2+y^2+z^2=1$ which lies in first quadrant.

Soln. Step 1 ~~[illegible]~~ Find $\vec{F}$.

$$\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$$

Step 2 find S (form of S) ($\hat{n}$) $= \frac{\text{grad}}{|\text{grad}|}$

S: part of surface of sphere $x^2+y^2+z^2=1$ which lies in first quadrant. $\rightarrow$ (Shape)

So, let $\phi = x^2+y^2+z^2-1$

Step 3 find $\hat{n}$. (on basis of step 2)

$$\hat{n} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} - \frac{\vec{n}}{|\vec{n}|}$$

$$\vec{n} = \text{grad}\,\phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-1)$$

$$= \hat{i}\frac{\partial(x^2+y^2+z^2-1)}{\partial x} + \hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2-1) + \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2-1)$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}.$$

$$\vec{n}\ \ \text{grad}\,\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\hat{n} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4(x^2+y^2+z^2)}} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{2}$$

$$\Rightarrow \boxed{\hat{n} = x\hat{i} + y\hat{j} + z\hat{k}}$$

Step 4 form projection (ds)

let R be projection of surface on xy-plane

i.e. $R: x^2+y^2=1 \quad [\because z=0]$

then, $$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{|(x\hat{i}+y\hat{j}+z\hat{k})\cdot\hat{k}|} = \frac{dx\,dy}{z}$$

$$\vec{F}\cdot\hat{n} = (yz\hat{i}+xz\hat{j}+yx\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= xyz + xyz + xyz$$

$$= 3xyz$$

$$\iint \vec{F}\cdot\hat{n}\,ds = \iint_S (3xyz)\left(\frac{dx\,dy}{z}\right)$$

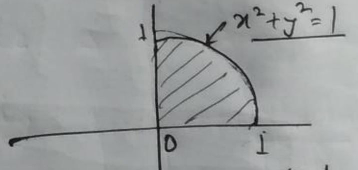
$$= \iint_R 3xy\,dx\,dy$$

Step 5 Apply Region R

Since, $R: x^2+y^2=1$ is circle in first quadrant

So use polar coordinate, $x = r\cos\theta$, $y = r\sin\theta$,

$dx\,dy = r\,dr\,d\theta$



$r = 0$ to 1

$\theta = 0$ to $\pi/2$

Step 6

$$\iint \vec{F}\cdot\hat{n}\,ds = \iint_R 3xy\,dx\,dy$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} 3.(r\cos\theta).(r\sin\theta).(r\,dr\,d\theta)$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} 3r^3 \sin\theta\cos\theta\,dr\,d\theta$$

$$= 3\int_{\theta=0}^{\pi/2}\left(\frac{r^4}{4}\right)_0^1 \sin\theta\cos\theta\,d\theta = \frac{3}{4}\int_{\theta=0}^{\pi/2}\frac{2\sin\theta\cos\theta\,d\theta}{2} = \frac{3}{8}\int_0^{\pi/2}\sin 2\theta\,d\theta$$

$$= \frac{3}{8}\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2} = \frac{-3}{16}(-1-1) = \frac{3}{8} \text{ Ans}$$

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VOLUME INTEGRAL

Volume integral integrates over three dimentional region. The volume integral of i) $\vec{F}$ over V is

$$\iiint \vec{F}\, dV = \hat{i}\iiint F_1\, dx\,dy\,dz + \hat{j}\iiint F_2\, dx\,dy\,dz + \hat{k}\iiint F_3\, dx\,dy\,dz$$

ii) ϕ over V is

$$\iiint \phi\, dV = \iiint \phi\, dx\,dy\,dz \quad \left(\because \phi = f_1(x,y,z)\right)$$

Example :- Evaluate $\int_V (2x+y)\,dV$, where V is closed region bounded by cylinder $z = 4 - x^2$ and planes $x=0$, $y=0$, $y=2$, and $z=0$.

Soln

$$\int_V (2x+y)\,dV = \iiint (2x+y)\,dx\,dy\,dz$$

Region $z = 4 - x^2$ $\quad x=0$

$y=0 \quad y=2$

$z=0$

For $z=0$, $x^2=4$

$\Rightarrow x=2$

z varies from 0 to $4-x^2$

y varies from 0 to 2

x varies from 0 to 2

$$\iiint (2x+y)\,dx\,dy\,dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{4-x^2} (2x+y)\,dx\,dy\,dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2} (2x+y)\,dx\,dy\,\left[z\right]_0^{4-x^2}$$

$$= \int_{x=0}^{2}\int_{y=0}^{2} (2x+y)(4-x^2)\,dx\,dy$$

$$= \int_{x=0}^{2}\int_{y=0}^{2} 2x(4-x^2)\,dx\,dy + \int_{x=0}^{2}\int_{y=0}^{2} y(4-x^2)\,dx\,dy$$

$$= \int_{x=0}^{2} 2x(4-x^2)\,dx \int_{y=0}^{2} dy + \int_{x=0}^{2} (4-x^2)\,dx \int_{y=0}^{2} y\,dy$$

$$= \left[-\frac{1}{2}(4-x^2)^2\right]_0^2 \left[y\right]_0^2 + \left[4x-\frac{x^3}{3}\right]_0^2 \left[\frac{y^2}{2}\right]_0^2$$

$$= 0-\left(-\frac{4^2}{2}\right)\times 2 + \left(8-\frac{8}{3}\right)\left(\frac{4}{2}\right)$$

$$= 16 + \frac{32}{3}$$

$$= \frac{80}{3} \text{ Ans}$$

GREEN'S THEOREM

Green's Theorem is relation between line integral and surface integral in xy-plane only.

If R is closed region in the xy- plane bounded by simple closed curve C (traversed in anti clockwise) and if $F_1(x,y)$ and $F_2(x,y)$ are continuous function having continuous partial derivatives in the region R, then

$$\int_C F_1(x,y)dx + F_2(x,y)dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

Note: $\vec{F} = F_1(x,y)\hat{\imath} + F_2(x,y)\hat{\jmath}$ *and* $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath}$, *Then*

$$\vec{F}.d\vec{r} = F_1(x,y)dx + F_2(x,y)dy$$

Question 1

Using Green's theorem, Evaluate $\int_C x^2y dx + y^3 dy$ where C is closed path formed by $y = x$ and $y = x^3$ from $(0,0)$ to $(1,1)$.

Soln. :

$$\int_C F_1(x,y)dx + F_2(x,y)dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

STEP I: Compare $\int_C F_1(x,y)dx + F_2(x,y)dy$ **with** $\int_C x^2y dx + y^3 dy$

$F_1(x,y) = x^2y$ **and** $F_2(x,y) = y^3$

STEP II:

$$\frac{\partial F_1}{\partial y} = x^2, \qquad \frac{\partial F_2}{\partial x} = 0$$

Soln. : STEP III: Substitute partial derivatives $\frac{\partial F_1}{\partial y} = x^2$, $\frac{\partial F_2}{\partial x} = 0$

$$\int_C F_1(x,y)dx + F_2(x,y)dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

$$\int_C x^2y dx + y^3 dy = \iint_R (0 - x^2) dxdy$$

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$$= \iint_R -x^2 dxdy$$

R is region bounded by $y = x$ and $y = x^3$

$$\int_C x^2 y dx + y^3 dy = \iint_R -x^2 dxdy$$

STEP IV: Find limit of y in terms of x from given Curves

R is region bounded by $y = x$ and $y = x^3$

Limit of y is from $y = x^3$ to $y = x$

STEP V: Find limit of x using given Curves

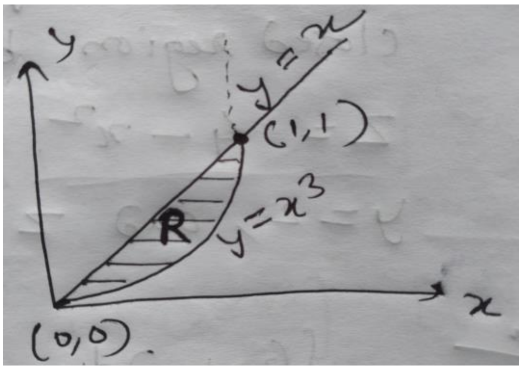
$y = x$ and $y = x^3$ intersect at

$x = x^3$

$x(1 - x^2) = 0 \Rightarrow x = 0, \quad x = 1$

So, Limit of x is from $x = 0, \ to \ x = 1$

Figure



STEP VI: Substitute limit in surface integral

$$\int_C x^2 y dx + y^3 dy = \iint_R -x^2 dxdy$$

$$= \int_{x=0}^{x=1} \int_{y=x^3}^{y=x} -x^2 dxdy = \int_{x=0}^{x=1} -x^2 dx \ [y]_{x^3}^{x}$$

$$= \int_{x=0}^{x=1} -x^2 \ (x - x^3) \ dx = -\left(\frac{x^4}{4} - \frac{x^6}{6}\right)_0^1$$

$$= \frac{1}{6} - \frac{1}{4} = \frac{2-3}{12} = -\frac{1}{12} \quad Ans.$$

<u>**Question 2**</u>

Using Green's theorem, Evaluate $\int_C (1 + xy^2)dx - x^2 y dy$ where C consist of arc of parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$.

Soln. :

$$\int_C F_1(x, y)dx + F_2(x, y)dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

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STEP I: Compare $\int_C F_1(x,y)dx + F_2(x,y)dy$ **with** $\int_C (1+xy^2)dx - x^2ydy$

$$F_1(x,y) = (1+xy^2) \qquad \text{and} \qquad F_2(x,y) = -x^2y$$

STEP II:

$$\frac{\partial F_1}{\partial y} = 2xy, \qquad \frac{\partial F_2}{\partial x} = -2xy$$

Soln. : STEP III: Substitute partial derivatives $\frac{\partial F_1}{\partial y} = 2xy,\quad \frac{\partial F_2}{\partial x} = -2xy$

$$\int_C F_1(x,y)dx + F_2(x,y)dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

$$\int_C x^2ydx + y^3dy = \iint_R (-2xy - 2xy)dxdy$$

$$= \iint_R -4xydxdy$$

C consist of arc of parabola $y = x^2$ **from** $(-1,1)$ **to** $(1,1)$.

$$\int_C (1+xy^2)dx - x^2ydy = \iint_R -4xydxdy$$

STEP IV: Find limit of y in terms of x from given Curves

R **is region bounded by** $y = x^2$ **and** $y = 1$

STEP V: Find limit of x using given Curves

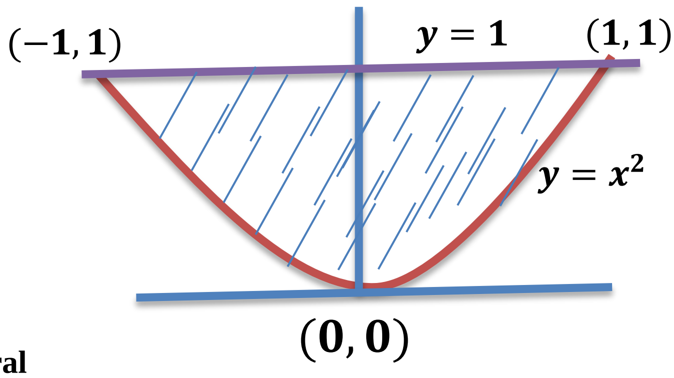
$y = x^2$ **and** $y = 1$ **intersect at**

$1 = x^2$

$x = -1, 1$

So, Limit of x *is from* $x = -1$ *to* $x = 1$

Limit of y **is from** $y = x^3$ **to** $y = x$



STEP VI: Substitute limit in surface integral

$$\int_C (1+xy^2)dx - x^2ydy = \iint_R -4xydxdy$$

$$= \int_{x=-1}^{x=1}\int_{y=x^2}^{y=1} -4xydxdy$$

$$= -4\int_{x=0}^{x=1} xdx\left[\frac{y^2}{2}\right]_{x^2}^{1}$$

$$= -2\int_{x=-1}^{x=1} x\,(1-x^4)\,dx \qquad = -2\left[\frac{x^2}{2}-\frac{x^6}{6}\right]_{-1}^{1} = 0$$

STOKE'S THEOREM

Stoke's Theorem is relation between line integral and surface integral.

If $\vec{F}$ is any continuous differentiable vector function and S is surface bounded by a curve C then,

$$\int_C \vec{F}.d\vec{r} = \iint_R Curl\vec{F}\ .\hat{n}\,ds$$

$$\int_C \vec{F}.d\vec{r} = \iint_R \left(\nabla\times\vec{F}\right)\ .\hat{n}\,ds$$

1. **The Green's theorem is known as Stokes Theorem in a plane.**

2. $\int_C F_1(x,y)dx + F_2(x,y)dy + F_3(x,y)dy = \iint_R\left[\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)dydz + \left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)dxdz + \left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)dxdy\right]$

IMPORTANT NOTES

1) **For a circle** $x^2+y^2=r^2$

Polar coordinates are $x = rcos\theta,\ y = rsin\theta\ \ dxdy = rdrd\theta$

Limit of r vary from 0 to r. Limit of θ vary from 0 to 2π (Depends on problem)

2) For a sphere $x^2+y^2+z^2=r^2$

Polar coordinates are

$$x = rsin\phi cos\theta,\ \ y = rsin\phi sin\theta,\ \ z = rsin\phi,\ \ \ dxdy = r^2 sin\phi drd\theta$$

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Limit of ϕ vary from 0 to π

3) For a parabola $y = x^2$

Parametric form are $x = t,\ y = t^2,\ dx = dt,\ dy = 2t dt$

4) For a Cylinder $x^2 + y^2 = r^2$ **,** $z = a$

cylindrical form is $x = r\cos\theta,\ \ y = r\sin\theta\ ,\ z = z$

5) **If the projection of surface S in xy plane then $\hat{n} = \hat{k}$**

Thus, $ds = dxdy$

6) Let S is the surface of shape ϕ , then normal vector

$$\vec{n} = grad\phi$$

$$\hat{n} = \frac{grad\phi}{|grad\phi|}$$

Question 1

Using Stoke's theorem, Evaluate $\int_C \vec{F}.d\vec{r}$ **where** $\vec{F} = y^2\hat{\imath} + xy\hat{\jmath} + xz\hat{k}$**, and C isbounding curve of hemisphere** $x^2 + y^2 + z^2 = 9$**, z > 0 oriented in a positive direction.**

Soln. : $\int_C \vec{F}.d\vec{r} = \iint_R (\nabla \times \vec{F})\ .\hat{n}\, ds$

STEP I: Find $\vec{F}$

$$\vec{F} = y^2\hat{\imath} + xy\hat{\jmath} + xz\hat{k}$$

STEP II: Obtain $\nabla \times \vec{F}$

$$Curl\,\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & xy & xz \end{vmatrix} = -z\hat{\jmath} - y\hat{k}$$

STEP III: Find $\hat{n}$

Let shape $\phi = x^2 + y^2 + z^2 - r^2$

$$grad\phi = \nabla\phi = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$$

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$$\hat{\boldsymbol{n}} = \frac{grad\phi}{|grad\phi|} = \frac{2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}}{\sqrt{4(x^2 + y^2 + z^2)}} = \frac{2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}}{2\ .\ 3}$$

$$\hat{\boldsymbol{n}} = \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{3}$$

STEP IV: Find ds $\qquad ds = \frac{dxdy}{|\hat{n}.\hat{k}|} = \frac{dxdy}{\left|\frac{x\hat{\imath}+y\hat{\jmath}+z\hat{k}}{3}.\hat{k}\right|} = 3\frac{dxdy}{z}$

STEP V: Substitute $\nabla \times \vec{F}$, $\hat{n}$ and ds in

$$\int_C \vec{F}.d\vec{r} = \iint_R \left(\nabla \times \vec{F}\right) .\hat{n}\, ds$$

$$= \iint_R \left(-z\hat{\jmath} - y\hat{k}\right) .\left(\frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{3}\right) 3\frac{dxdy}{z}$$

$$= \iint_R (-zy - yz).\ \frac{dxdy}{z} = -2\iint_R ydxdy$$

$$\int_C \vec{F}.d\vec{r} = -2\iint_R ydxdy$$

STEP VI: Take projection in xy-plane and find limit of x and y

Projection of given Surface S $x^2 + y^2 + z^2 = 9$ in xy-plane (z = 0) is

$$x^2 + y^2 += 9$$

Put $\quad x = rcos\theta,\ y = rsin\theta\ \ dxdy = rdrd\theta$

Limit of r vary from 0 to 3. Limit of θ vary from 0 to 2π

STEP VII: Apply limit of r and θ

$$\int_C \vec{F}.d\vec{r} = -2\iint_R ydxdy = -2\int_{\theta=0}^{2\pi}\int_{r=0}^{r=3} rsin\theta.\ rdrd\theta$$

$$= -2\int_{\theta=0}^{\theta=2\pi} sin\theta d\theta.\ \left[\frac{r^3}{3}\right]_0^3$$

$$= -18\int_{\theta=0}^{\theta=2\pi} sin\theta d\theta$$

$$= -18[-cos\theta]_0^{2\pi} = 0\ \ Ans.$$

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Question

Verify Stoke's theorem for the function $\vec{F} = y^2\hat{\imath} + xy\hat{\jmath}$ integrated along the square whose sides are $x = 0,\ y = 0,\ x = a,\ y = a$ in the plane $z = 0$.

Soln. :According to Stokes theorem

$$\int_C \vec{F}.d\vec{r} = \iint_R (\nabla \times \vec{F})\ .\hat{n}\ ds$$

R.H.S-STEP I:

$$\vec{F} = y^2\hat{\imath} + xy\hat{\jmath}$$

STEP II: Obtain $\nabla \times \vec{F}$

$$Curl\,\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & xy & 0 \end{vmatrix} = (y - 2y)\hat{k} = -y\hat{k}$$

$$\int_C \vec{F}.d\vec{r} = \iint_R (\nabla \times \vec{F})\ .\hat{n}\ ds$$

$\vec{\nabla} \times \vec{F} = -y\hat{k}$

STEP III: Find $\hat{n}$

Since the square is in $xy - plane$

Hence, $\hat{n} = \hat{k}$

STEP IV: Find ds

$ds = \frac{dxdy}{|\hat{n}.\hat{k}|} = dxdy \Rightarrow \quad ds = dxdy$

$$\int_C \vec{F}.d\vec{r} = \iint_R (\nabla \times \vec{F})\ .\hat{n}\ ds$$

STEP V- Limit of x and y

$$x = 0 \text{ to } x = a \text{ and } y = 0 \text{ to } y = a$$

R.H.S

$\iint_R (\nabla \times \vec{F})\ .\hat{n}\ ds = \int_{x=0}^{a}\int_{y=0}^{a} -y\hat{k}\ .\ \hat{k}dxdy$

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$$= -\int_{x=0}^{a}\int_{y=0}^{a} y\,dx\,dy = -\int_{x=0}^{a}\left[\frac{y^2}{2}\right]_0^a dx = -\frac{a^2}{2}[x]_0^a = -\frac{a^3}{2}$$

$$\iint_R (\nabla \times \vec{F})\,.\hat{n}\,ds = -\frac{a^3}{2}$$

L.H.S.

$$\int_C \vec{F}.d\vec{r} = \int_{C_1}\vec{F}.d\vec{r} + \int_{C_2}\vec{F}.d\vec{r}$$

$$+\int_{C_3}\vec{F}.d\vec{r} + \int_{C_4}\vec{F}.d\vec{r}$$

STEP I- $\vec{F} = y^2\hat{\imath} + xy\hat{\jmath}$

$d\vec{r} = dx\hat{\imath} + dy\hat{\jmath}$

$$\vec{F}.d\vec{r} = (y^2\hat{\imath} + xy\hat{\jmath}).(dx\hat{\imath} + dy\hat{\jmath}) = y^2dx + xydy$$

$$\vec{F}.d\vec{r} = y^2dx + xydy$$

Along C_1

$y = 0 \Rightarrow \; dy = 0$

$$\int_{C_1}\vec{F}.d\vec{r} = \int_{C_1} y^2dx + xydy$$

$$= \int_0^a 0.dx + x.0.0 = 0$$

$$\int_{C_1}\vec{F}.d\vec{r} = 0$$

Along C_2

$x = a \Rightarrow \; dx = 0$

Limit of y is from ***y = 0 to y = a***

$$\int_{C_2}\vec{F}.d\vec{r} = \int_{C_2} y^2dx + xydy$$

$$= \int_{C_2} y^2.0 + a.ydy = \int_0^a a.ydy \;\; a.\left[\frac{y^2}{2}\right]_0^a = \frac{a^3}{2}$$

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$$\int_{C_2} \vec{F}.d\vec{r} = \frac{a^3}{2}$$

Along C_3

$$y = a \Rightarrow \quad dy = 0$$

Limit of x is from *x = a to x = 0*

$$\int_{C_3} \vec{F}.d\vec{r} = \int_{C_3} y^2dx + xydy$$

$$= \int_{C_3} a^2.dx + xy.0 \ = a^2\int_a^0 dx = a^2(0-a) = -a^3$$

$$\int_{C_3} \vec{F}.d\vec{r} = -a^3$$

Along C_4

$$x = 0 \Rightarrow \quad dx = 0$$

$$\int_{C_4} \vec{F}.d\vec{r} = \int_{C_4} y^2dx + xydy$$

$$= \int_a^0 y^2.0 + 0.ydy = 0$$

$$\int_{C_4} \vec{F}.d\vec{r} = 0$$

L.H.S.

$$\int_{C} \vec{F}.d\vec{r} = \int_{C_1} \vec{F}.d\vec{r} + \int_{C_2} \vec{F}.d\vec{r} + \int_{C_3} \vec{F}.d\vec{r} + \int_{C_4} \vec{F}.d\vec{r}$$

$$= 0 + \frac{a^3}{2} - a^3 + 0$$

$$= -\frac{a^3}{2}$$

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$$\int_C \vec{F}.d\vec{r} = \iint_R (\nabla \times \vec{F}) \,.\hat{n}\, ds = -\frac{a^3}{2} \qquad \textbf{Hence Verified}.$$

DIVERGENCE THEOREM

Divergence Theorem is relation between surface integral and volume integral.

If $\vec{F}$ is any continuous differentiable vector function in region V bounded by a closed surface S then,

$$\iint_R \vec{F}\,.\hat{n}\,ds = \iiint_V div\vec{F}\,dV$$

$$\iint_R \vec{F}\,.\hat{n}\,ds = \iiint_V (\nabla.\vec{F})dV$$

1. **If $\vec{F}$ is solenoidal then $\iint_R \vec{F}\,.\hat{n}\,ds = 0$.**

2. $\iint_S F_1(x,y)dydz + F_2(x,y)dxdz + F_3(x,y)dxdy = \iiint \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dV$

Question 1

Using Divergence theorem theorem, Evaluate $\iint_R \vec{F}\,.d\vec{S}$ **where** $\vec{F} = 4x\hat{\imath} - 2y^2\hat{\jmath} + z^2\hat{k}$**, and S is surface bounding** $region\ x^2 + y^2 = 4$**, z = 0, z = 3.**

Soln. : $\iint_R \vec{F}\,.d\vec{S} = \iiint_V (\nabla.\vec{F})dV$

STEP I: Find $\vec{F}$

$$\vec{F} = 4x\hat{\imath} - 2y^2\hat{\jmath} + z^2\hat{k}$$

STEP II: Obtain $\nabla.\vec{F}$

$$div\ \vec{F} = \vec{\nabla}.\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial 4x}{\partial x} + \frac{\partial 2y^2}{\partial y} + \frac{\partial z^2}{\partial z} = 4 - 4y + 2z$$

STEP III: Substitute $\nabla.\vec{F} = 4 - 4y + 2z$ in Divergence theorem

$$\iint_R \vec{F}\,.d\vec{S} = \iiint_V (\nabla.\vec{F})dV$$

$$= \iiint_V (4 - 4y + 2z)dxdydz$$

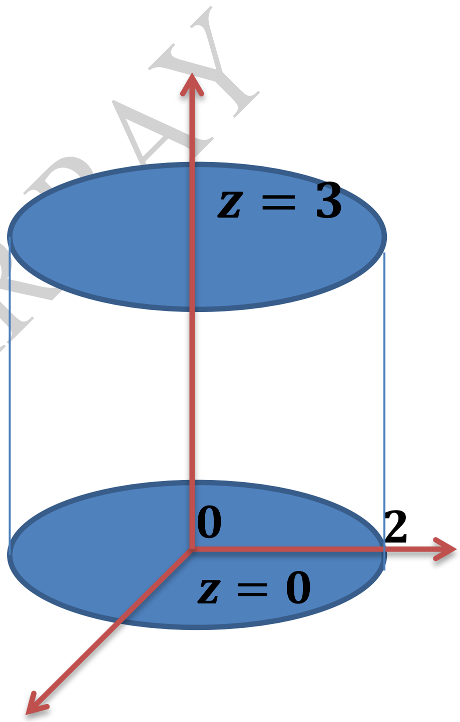
STEP IV: Find range of z from $region$.

Limit of z is from 0 to 3

$$\iint_R \vec{F} \cdot d\vec{S} = \iint_R \int_{z=0}^{z=3} (4 - 4y + 2z)dxdydz$$

$$= \iint_R (4z - 4yz + z^2)_0^3 dxdydz$$

$= \iint_R (21 - 12y)dxdy$



STEP V: Find limit of x and y

Surface S : $x^2 + y^2 = 4$ which is circle, so polar coordinates

$x = rcos\theta,\ y = rsin\theta\ \ dxdy = rdrd\theta$

STEP VI : Substitute limit

$x = rcos\theta,\ y = rsin\theta\ \ dxdy = rdrd\theta$

Limit of r vary from 0 to 2. Limit of θ vary from 0 to 2π

$\iint_R \vec{F} \cdot d\vec{S} = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (21 - 12rsin\theta).rdrd\theta$

$$= \int_{\theta=0}^{\theta=2\pi} \left(21\frac{r^2}{2} - 4r^3 sin\theta\right)_0^2 d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} (42 - 32sin\theta)d\theta$$

$$= (42\theta - 32cos\theta)_0^{2\pi}$$

$$= (42 \times 2\pi - 32(0 - 0)) = 84\pi$$

<u>Question 2</u>

Using Divergence theorem theorem, Evaluate $\iint_R \vec{F} \cdot d\vec{S}$ where $\vec{F} = x\hat{\imath} - y\hat{\jmath} + (z^2 - 1)\hat{k}$, and S is surface bounding $region$ $x^2 + y^2 = 4$, z = 0, z = 3.

$$\iint_R \vec{F} \cdot d\vec{S} = \iiint_V (\nabla . \vec{F})dV$$

Soln. STEP I: Find $\vec{F}$

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$$\vec{F} = x\hat{\imath} - y\hat{\jmath} + (z^2 - 1)\hat{k}$$

STEP II: Obtain $\nabla.\vec{F}$

$$div\ \vec{F} = \vec{\nabla}.\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial x}{\partial x} + \frac{\partial(-y)}{\partial y} + \frac{\partial(z^2 - 1)}{\partial z} = 1 - 1 + 2z = 2z$$

STEP III: Substitute $\nabla.\vec{F} = 2z$ in Divergence theorem

$$\iint_R \vec{F}\ .d\vec{S} = \iiint_V (\nabla.\vec{F})dV$$

$$= \iiint_V (2z)dxdydz$$

STEP IV: Find range of z from given $region$.

Limit of z is from 0 to 1

$$\iint_R \vec{F}\ .d\vec{S} = \iint_R \int_{z=0}^{z=1} (2z)dxdydz$$

$$= \iint_R \left(\frac{2z^2}{2}\right)_0^3 dxdy$$

$$= \iint_R dxdy = Area\ of\ Surface$$

Surface is circle $x^2 + y^2 = 4$, Area of the surface = πr^2

$$\iint_R \vec{F}\ .d\vec{S} = \iint_R dxdy = \pi 2^2 = 4\pi \quad Ans.$$

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- There will be separate question set for Assignments and Practice.
- All these questions are also explained in following YouTube Link https://www.youtube.com/channel/UCYg9RXUfbL1fdQhk1KdFsZQ?view_as=subscriber